

# There is no axiomatic system for the quantum theory

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Recently, [arXiv:0810.3134] is accepted and published. We derive an inequality with two settings as tests for the existence of the Bloch sphere in a spin-1/2 system. The probability theory of measurement outcome within the formalism of von Neumann projective measurement violates the inequality. Namely, we have to give up the existence of the Bloch sphere. Or, we have to give up the probability theory of measurement outcome within the formalism of von Neumann projective measurement. Hence it turns out that there is a contradiction in the Hilbert space formalism of the quantum theory, viz., there is no axiomatic system for the theory.

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## I. INTRODUCTION

Recently, [1] is accepted and published. As a famous physical theory, the quantum theory (cf. [2, 3, 4, 5, 6]) gives accurate and at times remarkably accurate numerical predictions. Much experimental data has been agreed with the quantum predictions for long time.

Here we aim to show that there is a contradiction in the Hilbert space formalism of the quantum theory. This implies that there is no axiomatic system for the theory.

In what follows, we derive an inequality with two settings as tests for the existence of the Bloch sphere, a proposition of the quantum theory, in a spin-1/2 system. The probability theory of measurement outcome within the formalism of von Neumann projective measurement violates the inequality. What we need is only one spin-1/2 state (a two-dimensional state). Therefore, there is a contradiction in the set of propositions of the quantum theory in a spin-1/2 system, viz., there is no axiomatic system for the quantum theory.

## II. NOTATION AND PREPARATIONS

Throughout this paper, we assume von Neumann projective measurement and we confine ourselves to the finite-dimensional and the discrete spectrum case. Let  $\mathbf{R}$  denote the reals where  $\pm\infty \notin \mathbf{R}$ . We assume every eigenvalue in this paper lies in  $\mathbf{R}$ . Further, we assume that every Hermitian operator is associated with a unique observable because we do not need to distinguish between them in this paper.

We assume the validity of the quantum theory and we would like to investigate if the probability theory of quantum measurement outcome is possible. Let  $\mathcal{O}$  be the space of Hermitian operators described in a finite-dimensional Hilbert space, and  $\mathcal{T}$  be the space of density operators described in the Hilbert space. Namely,  $\mathcal{T} = \{\psi | \psi \in \mathcal{O} \wedge \psi \geq 0 \wedge \text{Tr}[\psi] = 1\}$ . Now we define the notation  $\theta$  which represents one result of quantum measurement. Suppose that the measurement of a Hermitian operator  $A$  for a system in the state  $\psi$  yields a value  $\theta(A) \in \mathbf{R}$ . Let us consider the following propositions. Here,  $\chi_\Delta(x)$ , ( $x \in \mathbf{R}$ ) represents the characteristic function.  $\Delta$  is any subset of the reals  $\mathbf{R}$ .

*Proposition:* BSF (the Born statistical formula),

$$\text{Prob}(\Delta)_{\theta(A)}^\psi = \text{Tr}[\psi \chi_\Delta(A)]. \quad (1)$$

The whole symbol  $(\Delta)_{\theta(A)}^\psi$  is used to denote the proposition that  $\theta(A)$  lies in  $\Delta$  if the system is in the state  $\psi$ . And Prob denotes the probability that the proposition holds.

Let us consider a probability space  $(\Omega, \Sigma, \mu_\psi)$ , where  $\Omega$  is a nonempty space,  $\Sigma$  is a  $\sigma$ -algebra of subsets of  $\Omega$ , and  $\mu_\psi$  is a  $\sigma$ -additive normalized measure on  $\Sigma$  such that  $\mu_\psi(\Omega) = 1$ . The subscript  $\psi$  expresses that the probability measure is determined uniquely when the state  $\psi$  is specified.

Let us introduce measurable functions (random variables) onto  $\Omega$  ( $f : \Omega \mapsto \mathbf{R}$ ), which is written as  $f_A(\omega)$  for an operator  $A \in \mathcal{O}$ . Here  $\omega \in \Omega$ . We introduce appropriate notation.  $P(\omega) \simeq Q(\omega)$  means  $P(\omega) = Q(\omega)$  holds almost everywhere with respect to  $\mu_\psi$  in  $\Omega$ .

*Proposition:* P (the probability theory of measurement outcome),

Measurable function  $f_A(\omega)$  exists for every Hermitian operator  $A$  in  $\mathcal{O}$ .

*Proposition:* D (the probability distribution rule),

$$\mu_\psi(f_A^{-1}(\Delta)) = \text{Prob}(\Delta)_{\theta(A)}^\psi. \quad (2)$$

Now, we see the following:

*Lemma:*[7] Let  $S_A$  stand for the spectrum of the Hermitian operator  $A$ . If

$$\begin{aligned}\text{Tr}[\psi A] &= \sum_{y \in S_A} \text{Prob}(\{y\})_{\theta(A)}^\psi y, \\ E_\psi(A) &:= \int_{\omega \in \Omega} \mu_\psi(d\omega) f_A(\omega),\end{aligned}$$

then

$$P \wedge D \Rightarrow \text{Tr}[\psi A] = E_\psi(A). \quad (3)$$

*Proof:* Note

$$\begin{aligned}\omega \in f_A^{-1}(\{y\}) &\Leftrightarrow f_A(\omega) \in \{y\} \Leftrightarrow y = f_A(\omega), \\ \int_{\omega \in f_A^{-1}(\{y\})} \frac{\mu_\psi(d\omega)}{\mu_\psi(f_A^{-1}(\{y\}))} &= 1, \\ y \neq y' &\Rightarrow f_A^{-1}(\{y\}) \cap f_A^{-1}(\{y'\}) = \emptyset.\end{aligned} \quad (4)$$

Hence we have

$$\begin{aligned}\text{Tr}[\psi A] &= \sum_{y \in S_A} \text{Prob}(\{y\})_{\theta(A)}^\psi y = \sum_{y \in \mathbf{R}} \text{Prob}(\{y\})_{\theta(A)}^\psi y = \sum_{y \in \mathbf{R}} \mu_\psi(f_A^{-1}(\{y\})) y \\ &= \sum_{y \in \mathbf{R}} \mu_\psi(f_A^{-1}(\{y\})) y \times \int_{\omega \in f_A^{-1}(\{y\})} \frac{\mu_\psi(d\omega)}{\mu_\psi(f_A^{-1}(\{y\}))} \\ &= \sum_{y \in \mathbf{R}} \int_{\omega \in f_A^{-1}(\{y\})} \mu_\psi(f_A^{-1}(\{y\})) \times \frac{\mu_\psi(d\omega)}{\mu_\psi(f_A^{-1}(\{y\}))} f_A(\omega) \\ &= \int_{\omega \in \Omega} \mu_\psi(d\omega) f_A(\omega) = E_\psi(A).\end{aligned} \quad (5)$$

QED.

Thus, one may assume the probability measure  $\mu_\psi$  is chosen such that the following relation is valid:

$$\text{Tr}[\psi A] = \int_{\omega \in \Omega} \mu_\psi(d\omega) f_A(\omega) \quad (6)$$

for every Hermitian operator  $A$  in  $\mathcal{O}$ .

From BSF, P, and D, the possible value of  $f_A(\omega)$  takes eigenvalues of  $A$  almost everywhere with respect to  $\mu_\psi$  in  $\Omega$ . That is, we have the following theorem.

**Theorem:** (*The possible values of measurement outcome*):

Let  $S_A$  stand for the spectrum of the Hermitian operator  $A$ . For every quantum state described in a Hilbert space  $\mathcal{H}$ ,

$$\text{BSF} \wedge P \wedge D \Rightarrow f_A(\omega) \in S_A, \quad (\mu_\psi - a.e.). \quad (7)$$

### III. THERE IS NO AXIOMATIC SYSTEM FOR THE QUANTUM THEORY

Here, we shall show the following theorem.

**Theorem:**

$$[\text{The Bloch sphere exists.}] \wedge \text{BSF} \wedge P \wedge D \Rightarrow \perp. \quad (8)$$

*Proof:* Assume a spin-1/2 state  $\psi$ . Let  $\vec{\sigma}$  be  $(\sigma_x, \sigma_y, \sigma_z)$ , the vector of Pauli operator. The measurements (observables) on a spin-1/2 state of  $\vec{n} \cdot \vec{\sigma}$  are parameterized by a unit vector  $\vec{n}$  (direction along which the spin component is measured). Here,  $\cdot$  is the scalar product in  $\mathbf{R}^3$ .

One has a quantum expectation value  $E_{\text{QM}}$  as

$$E_{\text{QM}} \equiv \text{Tr}[\psi \vec{n}_k \cdot \vec{\sigma}], \quad k = 1, 2. \quad (9)$$

One has  $\vec{x} \equiv x^{(1)}$ ,  $\vec{y} \equiv x^{(2)}$ , and  $\vec{z} \equiv x^{(3)}$  which are the Cartesian axes relative to which spherical angles are measured. Let us write the two unit vectors in the plane defined by  $x^{(1)}$  and  $x^{(2)}$  in the following way:

$$\vec{n}_k = \sin \theta_k \vec{x}^{(1)} + \cos \theta_k \vec{x}^{(2)}. \quad (10)$$

Here, the angle  $\theta_k$  takes two values:

$$\theta_1 = 0, \quad \theta_2 = \frac{\pi}{2}. \quad (11)$$

We shall derive a necessary condition for the quantum expectation value for the system in a spin-1/2 state given in (9). Namely, we shall derive the value of the scalar product  $(E_{\text{QM}}, E_{\text{QM}})$  of the quantum expectation value,  $E_{\text{QM}}$  given in (9). We use decomposition (10). We introduce simplified notations as

$$T_i = \text{Tr}[\psi \vec{x}^{(i)} \cdot \vec{\sigma}] \quad (12)$$

and

$$(c_k^1, c_k^2) = (\sin \theta_k, \cos \theta_k). \quad (13)$$

Then, we have

$$(E_{\text{QM}}, E_{\text{QM}}) = \sum_{k=1}^2 \left( \sum_{i=1}^2 T_i c_k^i \right)^2 = \sum_{i=1}^2 T_i^2 \leq 1, \quad (14)$$

where we have used the orthogonality relation  $\sum_{k=1}^2 c_k^\alpha c_k^\beta = \delta_{\alpha,\beta}$ . From a proposition of the quantum theory, the Bloch sphere, the value of  $\sum_{i=1}^2 T_i^2$  is bounded as  $\sum_{i=1}^2 T_i^2 \leq 1$ . Clearly, the reason of the condition (14) is the Bloch sphere

$$\sum_{i=1}^3 (\text{Tr}[\psi \vec{x}^{(i)} \cdot \vec{\sigma}])^2 \leq 1. \quad (15)$$

Thus a violation of the inequality (14) implies a violation of the existence of the Bloch sphere (in a spin-1/2 system).

Let us assume BSF, P, and D hold. In this case, a quantum expectation value, which is the average of the results of projective measurements (based on a probability space) is given by

$$E_{\text{QM}} = \int_{\omega \in \Omega} \mu_\psi(d\omega) f_{\vec{n}}(\omega). \quad (16)$$

The possible values of  $f_{\vec{n}}(\omega)$  are  $\pm 1$  (in  $\hbar/2$  unit) almost everywhere with respect to  $\mu_\psi$  in  $\Omega$ .

We shall derive a necessary condition for the quantum expectation value obtained by the projective measurements on a spin-1/2 state given in (16). We shall derive the absolute value of the scalar product  $|(E_{\text{QM}}, E_{\text{QM}})|$  of the quantum expectation value,  $E_{\text{QM}}$  given in (16). One has

$$\begin{aligned} |(E_{\text{QM}}, E_{\text{QM}})| &= \left| \sum_{k=1}^2 \left( \int_{\omega \in \Omega} \mu_\psi(d\omega) f_{\vec{n}_k}(\omega) \times \int_{\omega' \in \Omega} \mu_\psi(d\omega') f_{\vec{n}_k}(\omega') \right) \right| \\ &= \left| \sum_{k=1}^2 \left( \int_{\omega \in \Omega} \mu_\psi(d\omega) \int_{\omega' \in \Omega} \mu_\psi(d\omega') f_{\vec{n}_k}(\omega) f_{\vec{n}_k}(\omega') \right) \right| \\ &\simeq \left| \sum_{k=1}^2 \left( \int_{\omega \in \Omega} \mu_\psi(d\omega) \int_{\omega' \in \Omega} \mu_\psi(d\omega') \right) \right| = 2. \end{aligned} \quad (17)$$

We have used the following, viz.,

$$\text{BSF} \wedge \text{P} \wedge \text{D} \Rightarrow f_{\vec{n}_k}(\omega) \in \{\pm 1\}, \quad (\mu_\psi - a.e.). \quad (18)$$

Therefore, one has the value (17) in contradiction to (14). As we have shown, it should be that  $(E_{\text{QM}}, E_{\text{QM}}) \leq 1$  if we accept the existence of the Bloch sphere. However, using BSF, P, and D, the probability theory of the results of von Neumann projective measurements violates the inequality since  $|(E_{\text{QM}}, E_{\text{QM}})| = 2$ . Namely, we have to give up, at least, one of propositions, the Bloch sphere, BSF, P, and D. QED.

#### IV. SUMMARY

In summary, the probability theory of the results of von Neumann projective measurements cannot allow the existence of the Bloch sphere. These quantum-theoretical propositions must contradict each other. Therefore there is a contradiction in the set of propositions of the quantum theory. Hence there is no axiomatic system for the quantum theory. Our result was obtained in a quantum system which is in a spin-1/2 state.

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